Linear Algebra - Midterm Exam

March 6, 2025 18.30-20.30

IMPORTANT:
At the end place your exam in the pile, **under the envelope** corresponding to your tutorial group
Do not cover the envelope with your exam, so that others can read the tutorial group number on the envelope
To avoid delays when placing the exams, make sure you **know your tutorial group number in advance** (see BS)
To avoid confusion in case exams get mixed up in the piles, write your name and student number on all pages
Number the pages of the exam sequentially
Exams not returned to the correct pile or pages without name/student number may not get graded

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Exam rules:
- You can have a "cheat sheet". This is an A4 paper written on one side (see above).
- You are NOT allowed to have books, course notes, homework assignments, etc., laptops, e-readers, tablets, telephones, etc.
- You can use a normal calculator (not a programmable/graphic one).
- Give a clear explanation of your answer and show any relevant computations.
- You get no points for a result without any calculation/explanation.
- If you use a different method than the one asked to solve a problem, you'll get non points even if your answer is right.

QUESTIONS:

1. A system of equations with 3 unknowns (x_1, x_2, x_3), with parameters λ_1 and λ_2, is given by

$$\begin{aligned} 2\,x_1 + x_2 - \;\; 3\,x_3 &= \lambda_1 \\ x_1 \qquad + \;\lambda_1\,x_3 &= -2 \\ x_1 + x_2 + \qquad\quad &= 3\,\lambda_2 \end{aligned}$$

   (a) $\boxed{0.6}$ Solve the system for $(\lambda_1, \lambda_2) = (2, 1)$. Clearly indicate the applied steps. (You can use any correct method that was taught in the course.)

   (b) $\boxed{0.4}$ Use the determinant to determine for which value(s) of λ_1 and/or λ_2 the system has exactly one solution.

   (c) $\boxed{0.6}$ Use row reduction to determine for which value(s) of λ_1 and/or λ_2 the system has no solutions.

   (Both for (b) and (c) state explicitly whetrher it is "or" or "and" whenever applicable.)

   (d) $\boxed{0.4}$ Consider a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= 0 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= 0 \end{aligned}$$

   where the $a_{ij}, i = 1, ..., 4; j = 1, ..., 4$ are given constants and $x_i, i = 1, ..., 4$ are unknowns. Is the system consistent? Explain why.

2. Given $A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 5 & 8 \\ 2 & 5 & 12 \end{bmatrix}$

   (a) $\boxed{0.5}$ Compute the adjoint matrix of A, $adj(A)$.

   (b) $\boxed{0.3}$ Is A non-singular?

   (c) $\boxed{0.3}$ If possible, depending on your answer to point (b), use $adj(A)$ to compute A^{-1}.

Please turn over

(d) $\boxed{0.3}$ Is $B = \begin{bmatrix} 1 & -2 & 3 & 0 & 0 & 0 \\ 2 & -5 & 10 & 0 & 0 & 0 \\ 2 & -5 & 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & -6 & 2 \end{bmatrix}$
invertible? Explain. (Notice that, if it is, you do not need to calculate the inverse yet.)

(e) $\boxed{0.3}$ Is $C = \begin{bmatrix} 3 & 4 & -5 & 0 & 0 & 0 \\ 10 & 12 & -15 & 0 & 0 & 0 \\ -2 & -3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & -3 & 1 \end{bmatrix}$
invertible? Explain. (Notice that, if it is, you do not need to calculate the inverse yet.)

(f) $\boxed{0.3}$ For B and C, use any valid method to calculate the inverse where possible.

3. Consider the subspace spanned by the set of vectors $U = \{-1 + 2x + 3x^2, x - x^2, x - 2x^2\}$ in P_3 (remember that P_3 is the vector space of polynomials of degree $n < 3$).

   (a) $\boxed{0.3}$ What is the dimension of this subspace? Explain.

   (b) $\boxed{0.3}$ It is stated that the set of vectors $W = \{x, x^2, 1 + x + x^2\}$ is a basis of that subspace. Explain why this is true.

   (c) $\boxed{1}$ Compute the transition matrix from the canonical basis, $E = \{1, x, x^2\}$ to the basis W.

   (d) $\boxed{0.4}$ Given a polynomial in the canonical basis, $p(x) = a + bx + cx^2$, write this polynomial in the basis W.

4. Which of the given sets of vectors are linearly independent? Explain in each case why.

   (a) $\boxed{0.8}$ $\{1, \sinh x, \cosh x\} \in C^2[0-1]$. (*Hint: In case you need it,* $\sinh x = \dfrac{e^x - e^{-x}}{2}$ *and* $\cosh x = \dfrac{e^x + e^{-x}}{2}$)

   (b) $\boxed{0.6}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

   (c) $\boxed{0.6}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

5. In the elections in the US, 80% of those who voted Republican (R) will vote again R in the next election, 10% will vote Democrat (D) in the next election, and 10% will vote Independent (I) in the next election. From those who voted D, 70% will vote again D in the next election, 20% will vote R in the next election, and 10% will vote I in the next election. From those who voted I, 40% will vote again I in the next election, 30% will vote D in the next election, and 30% will vote R in the next election.

   (a) $\boxed{0.8}$ Write the transition matrix for this problem in which the rows, from top to bottom, represent the fraction of the voters gained from the other parties or retained from their own party by, respectively, the D, R and I after an election, and the columns represent the fraction of the voters lost to the other parties or retained to their own party by, respectively, the D, R and I after an election.

   (b) $\boxed{1.2}$ In 2004 the voters were distributed as follows: D=48%, R=51%, I=1%. Calculate the fraction of voters per party in the following election, in 2008.

NOTE: Maximum points possible, $p = 10$. Grade, $g = 0.9p + 1$

Solutions

As stated in the exam rules, they get no points if they give the result without explanation or calculations, even of the result is correct

1. (a) I will use row reduction of the augmented matrix.

   Any correct method (given in the course) is also fine

   I can write the augmented matrix for $\lambda_1 = 2$ and $\lambda_2 = 1$ and row reduce it to find the solution:

   I do not give the steps but the students have to indicate what they did

$$\begin{bmatrix} 2 & 1 & -3 & 2 \\ 1 & 0 & 2 & -2 \\ 1 & 1 & 0 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -8/5 \\ 0 & 1 & 0 & 23/5 \\ 0 & 0 & 1 & -1/5 \end{bmatrix}$$

   0.6 points for solving the system of equations

   They need to either write the system in diagonal form (as I did above), or give the solution (as I did below). If they write the matrix in echelon (triangular) form but do not find the solution, they lose 0.3 points

   So the solution is $\mathbf{x} = \begin{bmatrix} -8/5 \\ 23/5 \\ -1/5 \end{bmatrix}$.

   (b) Using the determinant for a generic λ_1:

$$det(A) = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & \lambda_1 \\ 1 & 1 & 0 \end{vmatrix} = -\lambda_1 - 3.$$

   0.2 points for calculating the determinant correctly

   The system will have a single solution if $\det(A) \neq 0 \Leftrightarrow -\lambda_1 - 3 \neq 0 \Leftrightarrow \lambda_1 \neq -3$.

   0.1 points for stating (or using without stating it explicitly) that the system will have a single solution if $\det(A) \neq 0$

   So the system will have only 1 solution as long as $\lambda_1 \neq -3$. regardless of the value of λ_2.

   0.1 points for the value of λ_1 for which the system has a single solution

   They get no points if they use any other method (see the question)

   (c) Applying row operations $\begin{bmatrix} 2 & 1 & -3 & \lambda_1 \\ 1 & 0 & a & -2 \\ 1 & 1 & 0 & 3\lambda_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} (\lambda_1+3) & 0 & 0 & \lambda_1^2 - 3\lambda_1\lambda_2 - 6 \\ 0 & (\lambda_1+3) & 0 & -\lambda_1^2 + 6\lambda_1\lambda_2 + 9\lambda_2 + 6 \\ 0 & 0 & (\lambda_1+3) & -\lambda_1 + 3\lambda_2 - 2 \end{bmatrix}$.

   0.3 points for the row operations to reach echelon form

   From the last row:
   $(\lambda_1 + 3)x_2 = -\lambda_1 + 3\lambda_2 - 2$. If take $\lambda_1 = -3 \Rightarrow 0 = 3 + 3\lambda_2 - 2$, which will only be consistent if $\lambda_2 = -1/3$. So the system will have no solution (will be inconsistent) if $\lambda_1 = -3$ **and** $\lambda_2 \neq -1/3$. (Notice that you would get the same result if you used any of the rows, it is only that the last one is the easiest.)

   0.3 points for the solution

   They get no points if they use any other method (see the question)
   They lose the last 0.3 points if they do not state explicitly the **and** in the condition is

   (d) This is a homogeneous system, $A\mathbf{x} = \mathbf{0}$, and such system has always at least the trivial solution, so the system is always consistent.

   0.4 points for this or an equivalent statement

2. (a) The $adj(A)$ is the matrix with elements that are the cofactors of a_{ij} transposed.

   I do not write intermediate steps but the students have to explain how they get the adjoint matrix

$$adj(A) = \begin{pmatrix} 20 & 4 & 4 \\ 40 & 20 & 0 \\ -20 & -9 & 1 \end{pmatrix}$$

0.5 points for the adjoint matrix.

They lose 0.1 points for every **new** algebraic error

(b) I use the determinant (students can use other valid methods) because if $\det(A) \neq 0 \Leftrightarrow \exists A^{-1}$.

I do not show how I computed the determinant; the students have to explain it

$\det(A) = 20$ so A is non-singular.

0.3 points for this or similar statement

But they get no points if they do not show how they computed the determinant

They lose 0.1 points for every **new** algebraic error

(c) $A^{-1} = adj(A)/\det(A) = \begin{bmatrix} 1 & 1/5 & 1/5 \\ 2 & 1 & 0 \\ -1 & -9/20 & 1/20 \end{bmatrix}$.

0.3 points for the inverse

They still get the 0.3 points if they made an algebraic mistake in the previous points and their result is different than this one, but self-consistent with their own calculation

They get no points if they use any other method (see the question)

(d) B is a block matrix, so the determinant is the product of the determinants of the blocks. But $\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 0$ so no, B is not invertible.

0.2 points for finding that B is not invertible

0.1 points for the explanation of why B is not invertible

Since the question said to explain, they lose the 0.1 points if they do not explain anything

(e) C is a block matrix, so the determinant is again the product of the determinants of the blocks. $\begin{vmatrix} 3 & 4 & -5 \\ 10 & 12 & -15 \\ -2 & -3 & 4 \end{vmatrix} = -1$, $\det(7) = 7$, and $\begin{vmatrix} 3 & 2 \\ -3 & 1 \end{vmatrix} = 9$, so yes, C is invertible.

0.2 points for finding that C is invertible

0.1 points for the explanation of why C is invertible

Since the question said to explain, they lose the 0.1 points if they do not explain anything

(f) Calculate the inverse of the blocks of C (B is not invertible); I do not give the steps, but the students have to show how they got the inverses of the blocks:

$$C^{-1} = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 10 & -2 & 5 & 0 & 0 & 0 \\ 6 & -1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/9 & -2/9 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}.$$

0.3 points for finding the inverse of B (C is not invertible). The simplest way to find the inverse is probably the one I used, but they can use any valid method

They lose 0.1 points for every **new** algebraic error

3. (a) Calculate the Wronskian, $W\{-1+2x+3x^2, x-x^2, x-2x^2\} = 2 \neq 0$, so the 3 vectors are linearly independent, That means that the three vectors are a basis of the subspace that they themselves define and hence you can generate the whole subspace with only three vectors. Therefore the dimension is 3.

0.2 points for finding that the 3 vectors are linearly independent using the Wronskian or any other valid method given in the course

0.1 points for stating **explicitly** that the dimension of the subspace is 3

Since this was asked in the question explicitly, they lose the 0.1 points if they do not give the dimension

(b) Again calculate the Wronskian, $W\{x, x^2, 1 + x + x^2\} = 2 \neq 0$, so the three vectors are linearly independent. Since the the dimension of this subspace is 3 (see previous point) and these are 3 linearly independent vectors they generate a 3-dimension space and hence they are basis of the subspace.

0.2 points for finding that the 3 vectors are linearly independent using the Wronskian or any other valid method given in the course

0.1 points for explaining why the vectors are a basis of the subspace

Since this was asked in the question explicitly, they lose the 0.1 points if they do not explain why the given vectors are a basis

(c) I need to express the canonical basis in terms of W. I write the elements of W in between parentheses for clarity:

Here I divide the points following the procedure I chose to solve the problem. If the student gets the correct result using a different procedure, e.g. if they write the vectors of the basis W in terms of the vectors of the standard basis and they then invert the matrix (but they have to explain what they did!), they should get the full points

$$\begin{aligned} 1 &= a_1(x) + a_2(x^2) + a_3(1 + x + x^2) \\ x &= b_1(x) + b_2(x^2) + b_3(1 + x + x^2) \\ x^2 &= c_1(x) + c_2(x^2) + c_3(1 + x + x^2) \end{aligned}$$

0.3 points for these three equations (0.1 points for each)

and I need to find all the constants, $a_1, a_2, ...$, etc. For that I use that 2 polynomials are equal if and only if the coefficients in front of equal powers of x are equal, which yields:

$1 = a_3,\ 0 = a_1 + a_3,\ 0 = a_2 + a_3$
$0 = b_3,\ 1 = b_1 + b_3,\ 0 = b_2 + b_3$
$0 = c_3,\ 0 = c_1 + c_3,\ 1 = c_2 + c_3$

Each of these is a system of 3 equations with 3 unknowns. I can solve them (I do not give any step) to find:

$$\begin{aligned} a_1 &= -1,\ a_2 = -1,\ a_3 = 1 \\ b_1 &= 1,\ b_2 = 0,\ b_3 = 0 \\ c_1 &= 0,\ c_2 = 1,\ c_3 = 0 \end{aligned}$$

0.3 points for solving each system (0.1 points for each)

So the transition matrix from W to the canonical basis is $S = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

0.4 points for the transition matrix (but see my comment at the top)

Do not give the 0.4 points if they write the matrix transposed (which is a common mistake, apparently). Still give the 0.6 points from the previous steps

(d) Given $p(x) = a + bx + cx^2$ in the canonical basis, I can find the same polynomial in the basis W using:
$S \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b-a \\ c-a \\ a \end{bmatrix}$, such that in the new basis: $p(x) = (b-a)x + (c-a)x^2 + a(1 + x + x^2)$.

0.4 points for this

I can check this (they do not need to do this, but it helps to be sure you got it right) expanding the above : $p(x) = (b-a)x + (c-a)x^2 + a(1+x+x^2) = bx - ax + cx^2 - ax^2 + a + ax + ax^2 = a + bx + cx^2$.

They do not need to check, so they do not lose points if they don't

4. (a) Here I use the Wronskian, $W(1, \sinh x, \cosh x) = \begin{bmatrix} 1 & \sinh x & \cosh x \\ 0 & \cosh x & \sinh x \\ 0 & \sinh x & \cosh x \end{bmatrix} = \cosh^2 x - \sinh^2 x = 1 \neq 0$, so the answer is yes.

0.8 points for this. They can also write $c_1(1) + c_2(\sinh x) + c_3(\cosh x) = 0$, take 3 arbitrary values for x and get a system of 3 equations with 3 unknowns, the c_i, and see whether this system has only the trivial solution, etc., like was done in one of the HW. See also example 8 in §3.3 of the book.

0.1 points if they only say that they have to use the Wronskian, but do nothing, not even write it

(b) Given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, I calculate $\det[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ -3 & 1 & 2 \end{vmatrix} = -11 \neq 0$ so yes.

0.6 points for this. They get full points if they use any other valid method.

(c) Here I do not need to calculate anything (but it is okay if the students do) because in $\mathbb{R}^2$ more than 2 vectors cannot be are linearly independent, so the answer is no.

0.6 points for this. They get full points if they use any other valid method.

5. (a) The transition matrix is:

$$A = \begin{bmatrix} & D & R & I \\ D & 0.70 & 0.10 & 0.30 \\ R & 0.20 & 0.80 & 0.30 \\ I & 0.10 & 0.10 & 0.40 \end{bmatrix}$$

(It is fine also if they write percentages instead of fractions.)

0.8 points for the matrix

(b) The vector of voters in 2004 was: $\mathbf{v}_0 = \begin{bmatrix} D \\ R \\ I \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.51 \\ 0.01 \end{bmatrix}$.

0.4 points

The vector in the following election (2008) will therefore be:

$$\mathbf{v}_1 = A\mathbf{v}_0 = \begin{bmatrix} 0.70 & 0.10 & 0.30 \\ 0.20 & 0.80 & 0.30 \\ 0.10 & 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 0.48 \\ 0.51 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.390 \\ 0.507 \\ 0.103 \end{bmatrix}.$$

0.8 points